ALGEBRAIC STRUCTURES ON GENERALIZED STRINGS

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ABSTRACT. A garland based on a manifold P is a finite set of manifolds homeomorphic to P with some of them glued together at marked points. Fix a manifold M and consider a space $\mathcal{N}$ of all smooth mappings of garlands based on P into M. We construct operations $\bullet$ and $[-,-]$ on the bordism groups $\Omega_*(\mathcal{N})$ that give $\Omega_*(\mathcal{N})$ the natural graded commutative assosiative and graded Lie algebra structures. We also construct two auto-homomorphisms proj and lift of $\Omega_*(\mathcal{N})$ such that $\mathrm{proj}(\mathrm{lift}\,\alpha_1 \bullet \mathrm{lift}\,\alpha_2) = [\alpha_1, \alpha_2]$ for all $\alpha_1, \alpha_2 \in \Omega_*(\mathcal{N})$. If P is a boundary, then $\mathrm{proj} \circ \mathrm{lift} = 0$ and thus $\Delta^2 = 0$ for $\Delta = \mathrm{lift} \circ \mathrm{proj}$. We show that under certain conditions the operations Δ and $\bullet$ give rise to Batalin-Vilkoviski and Gerstenhaber algebra structures on $\Omega_*(\mathcal{N})$.

In a particular case when $P = S^1$, the algebra $\Omega_*(\mathcal{N})$ is related to the string-homology algebra constructed by Chas and Sullivan [4].

1. PRELIMINARIES

We work in the smooth category.

Given a topological space X, we define a *mark* $\mathfrak{m}$ on X to be a finite family $q_\gamma, \gamma \in \Gamma$, of points of X, the case $\Gamma = \varnothing$ is allowed. It can also happen that $q_\gamma = q_{\gamma'}$ for $\gamma \neq \gamma'$. The number of elements in a mark $\mathfrak{m}$ is denoted by $|\mathfrak{m}|$ or $|\Gamma|$.

Given a marked space $X = (X, \mathfrak{m})$, a *marked map* $f : X \to Y$ is a map such that $f(q_\gamma) = f(q_{\gamma'})$ for all $\gamma, \gamma' \in \Gamma$. Frequently, if we need to indicate the mark $\mathfrak{m}$ on X, we will write $f : (X, \mathfrak{m}) \to Y$ and say that f is an $\mathfrak{m}$-marked map.

For future goals, we assign to each mark $\mathfrak{m}$ its *grading* $g(\mathfrak{m}) \in \mathbb{N}$.

A multimark $\mathfrak{M}$ on a space X is just a finite family $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$ of marks on X. It can happen that the different marks can have common elements, or even $\mathfrak{m}_i = \mathfrak{m}_j$ for $i \neq j$. A multimarked map $f : (X, \mathfrak{M}) \to Y$ is just a map which is a marked map for each mark $\mathfrak{m} \in \mathfrak{M}$.

Fix a closed connected oriented manifold P of dimension n and a connected oriented manifold M of dimension m, and let $\mathcal{P}$ be the set of all smooth maps $P \to M$ in C^∞-topology. Fix $k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\}$ and $g_1, \ldots, g_l \in \mathbb{N}$. Consider a multimarked manifold $(P_1 \sqcup \cdots \sqcup P_k; \mathfrak{m}_1, \ldots, \mathfrak{m}_l)$ with $g(\mathfrak{m}_i) = g_i$ for $i = 1, \ldots, l$. Then the set $\mathcal{N}(k; l, g_1, \ldots, g_l)$ of all multimarked maps $(P_1 \sqcup \cdots \sqcup P_k; \mathfrak{m}_1, \ldots, \mathfrak{m}_l) \to$

M factorized by the action of the permutation group S_k can be regarded as a subspace of the space $\left(P^{\sum |\mathfrak{m}_i|} \times \mathcal{P}^k\right)/S_k$, and we equip $\mathcal{N}(k; l, g_1, \ldots, g_l)$ with the subspace topology.

Now, we set

$$\mathcal{N} = M \sqcup \coprod_{k=1}^{\infty}\coprod_{l=0}^{\infty} \coprod_{g_1,\ldots,g_l} \mathcal{N}(k; l, g_1, \ldots, g_l).$$

Here the first summand M can be treated as $\mathcal{N}(0; 0)$.

1.1. Remark. Given a multimarked manifold $(P \sqcup \cdots \sqcup P, \mathfrak{M})/S_k \in \mathcal{P}_k/S_k$, consider the following equivalence relation $\simeq$ on it: Two points $a, b \in P \sqcup \cdots \sqcup P$ are equivalent if there exists a mark $\mathfrak{m} \in \mathfrak{M}$ such that $a, b \in \mathfrak{m}$. We define a *garland* (*based on the manifold P*) of $(P \sqcup \cdots \sqcup P, \mathfrak{M})$ to be the quotient space $\left((P \sqcup \cdots \sqcup P)/\simeq)/S_k\right)$, see Figure 1.
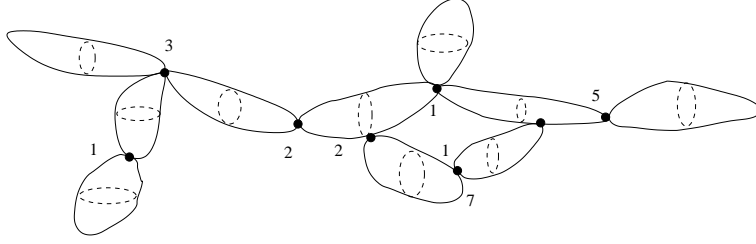

FIGURE 1. A garland with $\mathbb{N}$-graded marks

Clearly, every space $\mathcal{N}_k, k > 0$, consists of maps of garlands. In fact, garlands are precisely the objects we will work with. However, since garlands are not manifolds and therefore the problems with transversality can appear, we prefer to interpret garlands as multimarked manifolds and deal with the last ones.

Throughout the paper we work with oriented bordism theory $\Omega_*(-)$. All the necessary information can be found in books [12, 14, 15].

2. An operation $\bullet$ on $\mathcal{N}$

In this section we construct a commutative and associative operation

(2.1) $$\bullet : \Omega_i(\mathcal{N}) \otimes \Omega_j(\mathcal{N}) \to \Omega_{i+j-m}(\mathcal{N})$$

Let $\overline{\alpha}_1 : F_1^i \to \mathcal{N}$ and $\overline{\alpha}_2 : F_2^j \to \mathcal{N}$ be representatives of $\alpha_1 \in \Omega_i(\mathcal{N})$ and $\alpha_2 \in \Omega_j(\mathcal{N})$, respectively. Without loss of generality we assume that F_1^i, F_2^j are connected. So, we have adjoint maps $\omega_1 : F_1 \times N_1 \to M$

and $\omega_2 : F_2 \times N_2 \to M$. If, say, $\overline{\alpha}_2(F_2^j) \subset M = N(0;0) \subset \mathcal{N}$ then ω_2 is just $\overline{\alpha}_2 : F_2 \to M$.

Let $\mathfrak{m}_1, \mathfrak{m}_2$ be two marks of grading 1 on N_1 and N_2, respectively. Choose $q_1 \in \mathfrak{m}_1$ and $q_2 \in \mathfrak{m}_2$ and consider the pull-back diagram

(2.2)
$$\begin{array}{ccc} V_{q_1,q_2} & \xrightarrow{\ j_1\ } & F_1^i \times q_1 \\ \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle \omega_1} \\ F_2^j \times q_2 & \xrightarrow{\ \omega_2\ } & M \end{array}$$

(if $\overline{\alpha}_2(F_2^j) \subset M = N(0;0) \subset \mathcal{N}$ then $F_2^j \times q_2$ is just F_2^j). Using standard transversality arguments, we can and shall assume that $V = V_{q_1,q_2}$ is a smooth $(i + j - m)$-dimensional oriented manifold.

Together the maps

$$V \times N_1 \xrightarrow{\ j_1 \times \mathrm{id}\ } F_1 \times N_1 \xrightarrow{\ \omega_1\ } M$$

and

$$V \times N_2 \xrightarrow{\ j_2 \times \mathrm{id}\ } F_2 \times N_2 \xrightarrow{\ \omega_1\ } M$$

yield the map $\varphi : V \times (N_1 \sqcup N_2) \to M$. We equip $N_1 \sqcup N_2$ with the mark $\mathfrak{m}_1 \cup \mathfrak{m}_2$ and assign to this mark grading 1. All the other marks on N_1 and N_2 remain as they were and do not change their gradings. Clearly, for every $v \in V$, the map

$$N_1 \sqcup N_2 = \{v\} \times (N_1 \sqcup N_2) \xrightarrow{\ \varphi\ } M$$

is a marked map. So, the image of the adjoint to φ map $\psi = \psi_{q_1,q_2}$ belongs to $\mathcal{N}$, i.e. we have a map $\psi_{q_1,q_2} : V \to \mathcal{N}$.

It is easy to see that the bordism class of

$$\left[V = V_{\mathfrak{m}_1,\mathfrak{m}_2}, \psi = \psi_{\mathfrak{m}_1,\mathfrak{m}_2} \right]$$

depends only on $\mathfrak{m}_1, \mathfrak{m}_2$ (and not on the choices of $q_1 \in \mathfrak{m}_1, q_2 \in \mathfrak{m}_2$) and on the bordism classes α_1, α_2. Now, we define $\alpha_1 \bullet \alpha_2 = \left[V = V_{\mathfrak{m}_1,\mathfrak{m}_2}, \psi = \psi_{\mathfrak{m}_1,\mathfrak{m}_2} \right]$.

Put

(2.3)
$$\alpha_1 \bullet \alpha_2 = \sum_{\mathfrak{m}_1,\mathfrak{m}_2} [V_{\mathfrak{m}_1,\mathfrak{m}_2}, \psi_{\mathfrak{m}_1,\mathfrak{m}_2}] \in \Omega_{i+j-m}(\mathcal{N}),$$

where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ run over all marks of grading one on N_1 and N_2, respectively.

Following [4], we set $\mathbb{O}_i(\mathcal{N}) = \Omega_{i+m}(\mathcal{N})$. Now the pairing $\bullet$ gets the form

(2.4)
$$\bullet : \mathbb{O}_i(\mathcal{N}) \otimes \mathbb{O}_j(\mathcal{N}) \to \mathbb{O}_{i+j}(\mathcal{N})$$

We say that $|\alpha| = i$ whenever $\alpha \in \mathbb{O}_i(\mathcal{N})$. The following theorem follows directly from the definition of the operation $\bullet$.

2.1. Theorem. *The operation* $\bullet : \mathbb{O}_i(\mathcal{N}) \otimes \mathbb{O}_j(\mathcal{N}) \to \mathbb{O}_{i+j}(\mathcal{N})$, $i, j \in \mathbb{Z}$, *converts the abelian group* $\mathbb{O}_*(\mathcal{N})$ *into an associative graded commutative ring, i.e. the operation* $\bullet$ *has the following properties:*

1: $\alpha_1 \bullet \alpha_2 = (-1)^{|\alpha_1||\alpha_2|}\alpha_2 \bullet \alpha_1$, *for all* $\alpha_1, \alpha_2 \in \mathbb{O}_*(\mathcal{N})$;

2: $(\alpha_1 \bullet \alpha_2) \bullet \alpha_3 = \alpha_1 \bullet (\alpha_2 \bullet \alpha_3)$, *for all* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{O}_*(\mathcal{N})$;

3: $\alpha_1 \bullet (\alpha_2 + \alpha_3) = \alpha_1 \bullet \alpha_2 + \alpha_1 \bullet \alpha_3$ *for all* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{O}_*(\mathcal{N})$.

$\square$

We leave it to the reader to check that the unit of the operation $\bullet$ is given by the bordism class of the inclusion $M = \mathcal{N}(0; 0) \to \mathcal{N}$.

3. The string bracket $[-, -]$ on $\mathcal{N}$

Let all the notation be as in the previous section. Consider the pull-back diagram

$$
(3.1) \quad
\begin{array}{ccc}
W & \xrightarrow{\ j_1\ } & F_1^i \times N_1 \\
\downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle \omega_1} \\
F_2^j \times N_2 & \xrightarrow{\ \omega_2\ } & M
\end{array}
$$

Using standard transversality arguments, we can and shall assume that W is a smooth $(i + j + 2n - m)$-dimensional oriented manifold. Define the map

$$a_k : W \xrightarrow{\ j_k\ } F_k \times N_k \xrightarrow{\ p_2\ } N_k, k = 1, 2.$$

For every $w \in W$, we equip the manifold $N_1 \sqcup N_2$ with the mark $\mathfrak{m}_w = \{a_1(w), a_2(w)\}$ which, by definition, has grading 2. All the other marks on N_1 and N_2 remain as they were and do not change their gradings. Now, for every $w \in W$ we have a marked map $\psi_w : N_1 \sqcup N_2 \to M$ where, say, for $n \in N_1$ we have $\psi_w(n) = \omega_1(j_1(w), n)$. Furthermore, the correspondence $w \mapsto \psi_w$ gives us a map $\psi : W \to \mathcal{N}$.

It is easy to see that the bordism class of $[W, \psi]$ depends only on the bordism classes α_1, α_2.

3.1. Theorem. *The operation* $[-, -] : \mathbb{O}_i(\mathcal{N}) \otimes \mathbb{O}_j(\mathcal{N}) \to \mathbb{O}_{i+j+2n}(\mathcal{N})$, $i, j \in \mathbb{Z}$, *converts the abelian group* $\mathbb{O}_*(\mathcal{N})$ *into the graded Lie algebra, i.e. the operation* $[-, -]$ *has the following properties:*

1: $[\alpha_1, \alpha_2] = (-1)^{(|\alpha_1|+n)(|\alpha_2|+n)}[\alpha_2, \alpha_1]$, *for all* $\alpha_1, \alpha_2 \in \mathbb{O}_*(\mathcal{N})$;

2: $(-1)^?[[\alpha_1, \alpha_2], \alpha_3] + (-1)^?[[\alpha_2, \alpha_3], \alpha_1] + (-1)^?[[\alpha_3, \alpha_1], \alpha_2] = 0$, *for all* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{O}_*(\mathcal{N})$;

3: $[\alpha_1, \alpha_2 + \alpha_3] = [\alpha_1, \alpha_2] + [\alpha_1, \alpha_3]$, *for all* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{O}_*(\mathcal{N})$.

Proof. We are not able to put proper signs in the proposed Jakobi identity (2). (Both authors are sure that such identity exists, but up to present the authors get different answers.) However, we can prove the Jacobi identity modulo 2, and we do it now. Without loss of generality we assume that that the bordisms $\alpha_1, \alpha_2, \alpha_3$ have adjoint maps of the form $\overline{\alpha}_i : F_i \times P_i \to M$, $i = 1, 2, 3$, with F_i connected. Then all the garlands that appear when one opens the Lie brackets in the Jacobi identity consist of P_1, P_2, and P_3; and all garlands have two two-point marks that attach two out of three manifolds P_1, P_2 and P_3 to the third one. The appearing types of garlands subdivide in three different types depending on which one of P_1, P_2 and P_3 contains two out of four points of two two-point marks. (This is the manifold to which the other two are attached.)

The garlands where two out of four double points are on P_1 appear from the first and the third brackets of the Jacobi identity. The manifolds S_1 and S_3 describing the corresponding bordisms in $\mathcal{N}$ coming from the first and the third bracket are described by the following set-theoretic conditions.

$$S_1 = \left\{ (f_1, n_1^1, f_2, n_2, n_1^2, f_3, n_3) \right\} \subset F_1 \times N_1 \times F_2 \times N_2 \times N_1 \times F_3 \times N_3$$

such that $\overline{\alpha}_1(f_1, n_1^1) = \overline{\alpha}_2(f_2, n_2)$ and $\overline{\alpha}_1(f_1, n_1^2) = \overline{\alpha}_3(f_3, n_3)$.

$$S_3 = \left\{ (f_3, n_3, f_1, n_1^2, n_1, f_2, n_2) \right\} \subset F_3 \times N_3 \times F_1 \times N_1 \times N_1 \times F_2 \times N_2$$

such that $\overline{\alpha}_3(f_3, n_3) = \overline{\alpha}_1(f_1, n_1^2)$ and $\overline{\alpha}_1(f_1, n_1^1) = \overline{\alpha}_2(f_2, n_2)$.

It is clear that S_1 and S_2 can be transformed to each other via a coordinate permutation. Since the mappings of the corresponding garlands are determined by the values of f_1, f_2, f_3, we get that the mappings of garlands appearing from the corresponding points of S_1 and S_3 are the same. Thus this type of garlands in the Jacobi identity cancels out when one considers bordisms with $\mathbb{Z}_2$ coefficients.

The proof of the fact that the other two types of garlands appearing from the Jacobi identity cancel out is the same. The proof in the case where the garlands corresponding to α_1, α_2, and α_3 consist of more than one copy of the manifold P is obtained in the same way since the Lie bracket $[-, -]$ is defined via pull-backs over pairs of participating copies of P. $\qquad\square$

4. The homomorphisms lift, proj, and Δ

Consider a map $\varphi : F \to \mathcal{N}$ with F connected, and let $\overline{\varphi} : F \times N \to M$ be the adjoint marked map. Given a point $(f, n) \in F \times N$, we define a map $\psi_{f,n} : (N, \mathfrak{m}) \to M$ as follows: $\mathfrak{m} = \{n\}$ and has grading 1, and $\psi_{f,n}(n') = \overline{\varphi}(f, n')$ for all $n' \in N$. All the other marks increase their grading by 1. So, $\psi_{f,n} \in \mathcal{N}$, and we define $\psi : F \times N \to \mathcal{N}$ as $\psi(f, n) = \psi_{f,n}$. If we have a map $\varphi : F \to M = \mathcal{N}(0; 0) \subset \mathcal{N}$, we define $\psi : F \times P \to M$ to be the composition of φ and the projection $F \times P \to P$.

Now, the correspondence $\varphi \mapsto \psi$ yields a homomorphism

(4.1) $$\mathrm{lift} : \mathbb{O}_i(\mathcal{N}) \to \mathbb{O}_{i+n}(\mathcal{N}).$$

4.1. Definition. we define the homomorphism $\mathrm{proj} : \mathbb{O}_i(\mathcal{N}) \to \mathbb{O}_i(\mathcal{N})$ as follows. Given a map $\overline{\alpha} : F \to \mathcal{N}$ with the adjoint map $\omega : F \times N \to N$, the map proj just changes the marks on N as follows:

    1: if $g(\mathfrak{m}) \neq 1$ then proj decreases grading by 1;
    2: if $g(\mathfrak{m}) = 1$ and $|\mathfrak{m}| > 1$ then proj increases grading by 1;
    3: if $g(\mathfrak{m}) = 1$ and $|\mathfrak{m}| = 1$ then proj erases the mark.

Using the definitions of $\mathrm{proj}, \mathrm{lift}, \bullet$ and $[-, -]$ one gets the following result.

4.2. Proposition. *For all $\alpha_1, \alpha_2 \in \Omega_*(\mathcal{N})$ we have*

$$[\alpha_1, \alpha_2] = \mathrm{proj}(\mathrm{lift}(\alpha_1) \bullet \mathrm{lift}(\alpha_2)).$$

4.3. Proposition. *If the manifold P is a boundary, then $\mathrm{proj} \circ \mathrm{lift} \equiv 0$.*

Proof. Let $\alpha \in \mathbb{O}_i(\mathcal{N})$ be represented by a map $\overline{\alpha} : F \to \mathcal{N}$. It is easy to see that the element $\mathrm{proj} \circ \mathrm{lift}(\alpha)$ is represented by the map

$$F \times N \xrightarrow{\mathrm{projection}} F \xrightarrow{\omega} \mathcal{N}.$$

But N is boundary since P is, and thus the above map is zero-bordant. $\square$

4.4. Definition. We define the homomorphism $\Delta : \mathbb{O}_i(\mathcal{N}) \to \mathbb{O}_{i+n}(\mathcal{N})$ by setting $\Delta = \mathrm{lift} \circ \mathrm{proj}$.

4.5. Corollary. *If the manifold P is a boundary, then $\Delta^2 = 0$.*

Proof. This follows from Proposition 4.3, because

$$(\mathrm{lift} \circ \mathrm{proj}) \circ (\mathrm{lift} \circ \mathrm{proj}) = \mathrm{lift} \circ (\mathrm{proj} \circ \mathrm{lift}) \circ \mathrm{proj} = 0.$$

$\square$

5. Batalin–Vilkovyski and Gerstenhaber structures

Given a graded commutative algebra $A = (A, +, \bullet)$, consider an additive homogeneous homomorphism $\Delta : A_* \to A_{*+n}, n > 0$. A quadruple $A, +, \bullet, \Delta)$ is called a *Batalin–Vilkovyski algebra of degree n* if $\Delta^2 = 0$ and (cf. Getzler [8]).

$$
\begin{aligned}
\Delta(a \bullet b \bullet c) = \Delta(a \bullet b) \bullet c &+ (-1)^{|a|n} a \bullet \Delta(b \bullet c) \\
(5.1) \qquad\qquad &+ (-1)^{(|a|+n)|b|} b \bullet \Delta(a \bullet c) - \Delta(a) \bullet b \bullet c \\
&- (-1)^{|a|n} a \bullet \Delta(b) \bullet c - (-1)^{n(|a|+|b|)} a \bullet b \bullet \Delta(c)
\end{aligned}
$$

A *graded Gerstenhaber algebra of degree n* (called also *Poisson algebra* or *braid algebra* of degree n) is a graded commutative algebra V together with an operation $\{-, -\} : V \otimes V \to V$ satisfying the following relations (cf. Gerstenhaber [7]):

$$
(5.2) \qquad\qquad \{a, b\} = (-1)^n (-1)^{(|a|+n)(|b|+n)} \{b, a\}
$$

$$
(5.3) \qquad \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)} \{b, \{a, c\}\},
$$

and

$$
(5.4) \qquad\qquad \{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{(|a|+n)b} b \bullet \{a, c\}
$$

5.1. Proposition. *Let $(A, \bullet, \Delta)$ be a graded Batalin–Vilkovyski algebra. Set*

$$
\{a, b\} = (-1)^{|a|n} \Delta(a \bullet b) - (-1)^{|a|n} \Delta(a) \bullet b - a \bullet \Delta(b)
$$

Then the quadruple $(A, +, \bullet, \{-, -\})$ is a graded Gerstenhaber algebra of degree n.

Furthermore, the Leibnitz rule

$$
(5.5) \qquad \Delta\{a, b\} = (-1)^{n+1} \left(\{\Delta a, b\} + (-1)^{|a|n+1} \{a, \Delta b\} \right)
$$

holds.

Proof. Getzler [8, Prop. 1.2] proved these equalities for $n = 1$ (cf. also Penkava–Schwarz [11]). His approach works for arbitrary n, since we properly defined the signs as functions of n. $\qquad\square$

We hope but cannot prove in general that Δ satisfies relation (5.1). However, we have several examples where Δ defined as in Corollary 4.5, does satisfy (5.1), and thus $\mathbb{O}_*(\mathcal{N})$ possesses the structures of Batalin–Vilkovyski and Gerstenhaber algebras.

6. Comparison with Chas–Sullivan string theory

Here we just notice that for $P = S^1$ our theory is parallel to Chas-Sullivan string theory [4].

In [4] Chas and Sullivan consider the space $LM = \Lambda M/SO(2)$ where ΛM is the space of smooth free loops on M (i.e., maps $S^1 \to M$) and the $SO(2)$-action on ΛM is induced by the tautological $SO(2)$-action on $S^1 = SO(2)$. Furthermore, they define operations $\bullet$ and $[-,-]$ on $H_*(LM)$. Certainly, these operations can also be defined for bordisms instead of homology, and we get the operations

$$\bullet : \mathbb{O}_i(LM) \otimes \mathbb{O}_j(LM) \to \mathbb{O}_{i+j}(LM)$$

and

$$[-,-] : \mathbb{O}_i(LM) \otimes \mathbb{O}_j(LM) \to \mathbb{O}_{i+j+2}(LM).$$

6.1. **Remark.** In fact, we prefer to work with bordisms because it is a more geometric theory, it is easy to control transversality, etc. It seems, however, that our approach works for homology also, if we will be able to work properly with transversality.

Some troubles appear if one tries to generalize Chas–Sullivan construction for an arbitrary manifold P instead of S^1. There are several of them, but we discuss the following which looks most important. Chas–Sullivan use the co-H-space structure on S^1; this allow them to regard, say, a wedge (composition) of loops as a loop. However, in the class of closed manifold only homotopy spheres are co-H-spaces. So, if we want to consider any P instead of S^1 then we should, roughly speaking, convert P into a co-H-space.

It is instructive to consider an algebraic analog of the situation. If we have an abelian group A and want "to convert it to a ring", we consider the tensor algebra $T(A)$. So, conceptually, for a manifold P a role of an analog of the tensor algebra can play a space $P \sqcup P \vee P \sqcup \dots$ (the union of garlands). This is not a manifold, but we can resolve the singular points of wedges via considering the marked maps. This ideas lead us to spaces $\mathcal{N}$, which can be considered as analogs of ΛM. It is also worthy to mention that Chas–Sullivan uses $SO(2)$-action in order to work with wedges of non-pointed spaces. For arbitrary P, this problem is solved by using marked maps.

We finish this notes with a vague remark that a map of connected garland based on S^1 can be regarded as a map of circle. We do not dwell and explain here in detail the parallelism between our theory (for

$P = S^1$) and Chas–Sullivan one, but we hope to do it somewhere else (or in later version of this manuscript).

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